Teaching complicated mathematics at the university

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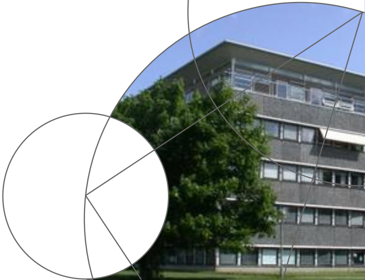

Overview of the talk

1. Type and aim of courses
2. General approach
3. Activation of students
4. Conclusion

Type of courses

Theoretical courses in probability theory (VidSand2 and Beting):

- Decomposition of measures
- Conditional expectations and conditional distributions
- Martingale theory: Optional sampling, convergence, CLT
- Brownian motion
- Conditional independence
- Markov chains: Markov property, ergodic theorem

Aim of courses

From course descriptions:

- "describe and prove the results on decomposition of signed measures"
- "derive and describe the main results on martingales"
- "discuss the relation between decomposition of measures and conditional expectations"
- "Give an oral presentation of a specific topic within the theory covered by the course"
- "Discuss the relation between conditional expectations and conditional distributions"

Aim of courses – continued

Theory and proofs play an important part of the courses:

- Adds to the deep understanding of the subject
- Proof techniques is a useful skill that has to be learned
- Simply very beautiful mathematics – students should see at least some of it!

General approach to lectures

Mix of blackboard and slides/computer

- Use slides to present definitions, theorems and small examples: Saves some time compared to blackboard
- Use blackboard to proofs and longer examples – the lecture is slowed down: Arguments are easier to follow
- Use computer simulations, when something can be illustrated (and I have the time)

General approach to lectures – continued

Use of enthusiasm and energy:

- As a lecturer, you can use energy and enthusiasm to keep students awake and interested.
- In particular variation in enthusiasm: Think about what is the most interesting/fascinating part of the lecture?

Activation of students

Reasons to activate students during a lecture:

- Hard to stay concentrated and focused for 45 minutes
- Learning is easier if the students can relate to the problem. E.g think about small problems by themselves.
- Deeper learning if the students are part of the development process (or feel they are)
- Creates interaction between students and lecturer. Makes adjustments easier.

Think about:

- Should students be forced to be active?

Downside

- You put yourself in a vulnerable situation – socially awkward to ask a question and not get an answer. Much easier not to involve the students.
- Student activities take up much time: Just explaining something would be much faster.
- Student activities make planning more difficult:
    - You don't know how much time an activity takes.
    - If you realise that something was not understood, then you have to react.

Questions

A simple activity is just to ask a lot of questions.

Problems:

- At most one student answers
- Panic among students!
- Awkward if no-one answers

Instead:

- Use of slides with questions and exercises (examples will follow)
- Give the students time to think and discuss.
- If many students: Make them vote.

Activities developing intuition

Working with the intuition can be useful:

- Good as introduction: Easier to understand complicated theory if you have some intuition
- Helpful to have an intuitive reference later on, when the result is used again

Challenge:

- My intuition may be different from the way the students think
- That leads to confusion

Extension of Tonelli's Theorem

Theorem (Theorem 1.3.2, Extended Tonelli)

Let μ be a probability measure on $(\mathcal{X}, \mathbb{E})$, and assume that $(P_x)_{x \in \mathcal{X}}$ is a $(\mathcal{X}, \mathbb{E})$–Markov kernel on $(\mathcal{Y}, \mathbb{K})$. Let λ be the integration of $(P_x)_{x \in \mathcal{X}}$ with respect to μ. For every $\mathbb{E} \otimes \mathbb{K}$–measurable function $f : \mathcal{X} \times \mathcal{Y} \to [0, \infty]$ it holds that

$$\int f(x,y) \, \mathrm{d}\lambda(x,y) = \iint f(x,y) \, \mathrm{d}P_x(y) \, \mathrm{d}\mu(x)$$

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Discussion: Compare with example 1.2.3 and discuss why this theorem is called an extension of Tonelli!

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Measures like λ can be seen as a generalisation of product measures $\mu\otimes\nu$ on $\mathcal{X}\times\mathcal{Y}$.

We will show that

$$\frac{1}{\sqrt{n}} M_n \xrightarrow{\mathcal{D}} \mathcal{N}(0,1)$$

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Discuss in 2 minutes:

- What will a plot of $(\frac{1}{n} M_n)_{n \geq 1}$ look like?
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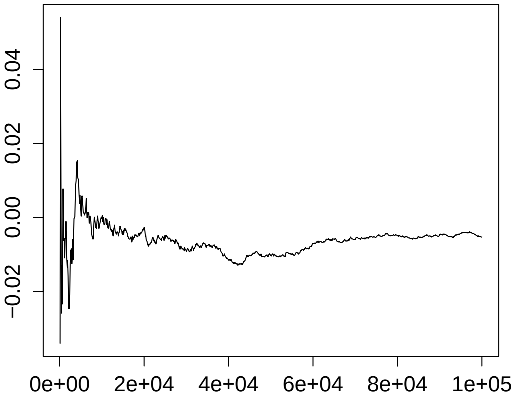
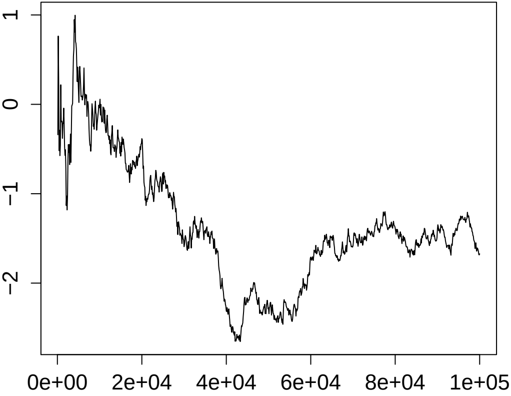


Figure: Simulation of $(\frac{1}{n}M_n)_{n\geq 1}$



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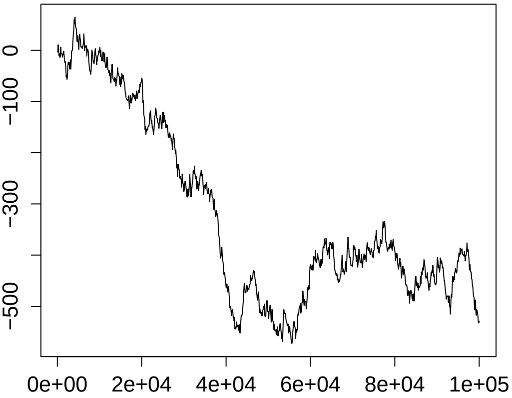
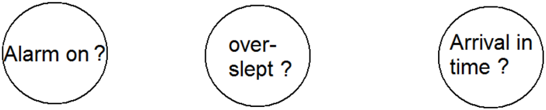


Figure: Simulation of $(M_n)_{n\geq 1}$

Discussion about your morning situation

Consider the following 3 events you could experience in the morning



- Do we have ordinary independence here?
- Try to identify a conditional independence between two events given a third
- Explain this using the asymmetric theorem

Relating to and combining known theory

Student activities can be used to remind students of material they (in principle) already know:

- You get to check whether the students have understood previous lectures
- Repeating helps understanding
- Remind students of some known theory that is about to be applied

Discussion – Memory game:-)

Consider the Markov kernels

$$1: \quad P^{*k} \qquad \text{and} \qquad 2: \qquad \mathcal{P}^k.$$

Combine them with the following conditional distributions, where $X_0, X_1, \ldots$ is a time homogeneous Markov chain with transition prob. $(P_x)_{x \in \mathcal{X}}$:

a) $(X_1, \ldots, X_k) \,|\, X_0$

b) $X_{n+k} \,|\, X_n$

c) $(X_n, \ldots, X_{n+k-1}) \,|\, X_{n-1}$

d) $X_k \,|\, X_0$

e) $X_{n+k} \,|\, X_0, \ldots, X_n$

f) $(X_n, \ldots, X_{n+k-1}) \,|\, X_0, \ldots, X_{n-1}$

Markov chains so far

Think about the arguments and the order:

- we defined a Markov chain
- we derived the finite–dimensional distributions for a Markov chain (if it exists)
- we argued that a probability measure exists on $(\mathcal{X}^\infty, \mathbb{E}^\infty)$ with finite–dimensional distributions like this
- we went back and saw that a process with this distribution IS a Markov chain

Think about: Could we have changed the order?

Involvement in proofs

- A proof could easily last 45 minutes: Breaks may be needed to avoid one–man lecturer shows and sleeping students!
- Can use activities to point out/repeat an important step - change of speed.
- Although the proof takes place on the blackboard, I (again) use slides to formulate questions/exercises.

First steps of proof of 5.5.1

Let $B = \{x \in \mathcal{X} \ : \ V(x) \leq r\}$ and

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Discuss: Why will the proof in that case be complete?

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Proof.

On the blackboard! □

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Discuss: Why can we conclude from the proof that

$$X_{\tau+1} \,|\, X_\tau \quad \overset{\mathcal{D}}{=} \quad X_{n+1} \,|\, X_n \ ?$$

Theorem (Theorem 3.4.3)

Let $\mathbb{A}$, $\mathbb{B}$, $\mathbb{G}$ and $\mathbb{H}$ be σ-algebras. It holds that

$$\mathbb{A} \perp\!\!\!\perp \mathbb{B} \,|\, \mathbb{H} \text{ and } \mathbb{A} \perp\!\!\!\perp \mathbb{G} \,|\, \mathbb{B} \vee \mathbb{H} \quad \Rightarrow \quad \mathbb{A} \perp\!\!\!\perp (\mathbb{B} \vee \mathbb{G}) \,|\, \mathbb{H}.$$

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Proof.

We show the asymmetric characterisation: Let $A \in \mathbb{A}$, then

$$P\big(A \mid (\mathbb{B} \vee \mathbb{G}) \vee \mathbb{H}\big) = P(A \mid \mathbb{B} \vee \mathbb{H}) = P(A \mid \mathbb{H}) \text{ a.s.}$$

Discuss: How is this equality obtained? □

Exercises

Activities could be small exercises:

- A way to give examples of applications of the theory
- As a lecturer you can check, whether a definition or result is understood

Potential problem:

- Party–killer if the exercise is too hard
- The exercise can/should be VERY simple: Should be possible to solve for most students within 2-3 minutes
- If the answer is very simple, the students should answer by voting

3–minute exercise:

Assume that $(X_n, \mathcal{F}_n)$ is adapted and that τ is a stopping time.

- Is X_4 $\mathcal{F}_6$–measurable?
- Is X_6 $\mathcal{F}_4$–measurable?
- Is $X_{\tau+1}$ $\mathcal{F}_\tau$–measurable?

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Example (/Exercise Continued)

Explain the calculation

$$E(S_{n+1}|\mathcal{F}_n) = S_n + E(Y_{n+1}) = S_n + (2p-1)$$

and decide the type of $(S_n, \mathcal{F}_n)$:

- ?-martingale, if $p > \frac{1}{2}$
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- ?-martingale, if $p < \frac{1}{2}$ (superMG)

Example 5.5.3: ARCH(1)–process

Define $X_0, X_1, X_2, \dots$ recursively by

$$X_{n+1} = \sqrt{(\gamma + \alpha X_n^2)}\epsilon_{n+1},$$

where (again) $\epsilon_1, \epsilon_2, \dots$ are iid $\mathcal{N}(0,1)$ (independent of X_0).

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Discuss:

- Why is this a Markov chain?
- How do you expect a simulation of $X_0, X_1, X_2, \ldots$ to look like?
- Which values (big/small) of α and γ do you think will make the Markov chain asymptotically stable?

Final comments – does it make a difference?

- No proof that it actually makes a difference, but I believe so...
- Students have seemed happy.
- Lecturing is more fun this way.
- Does not work without nice students:-)

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